Ramanujan School of Mathematics

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1. If $f(0) = 0$, we are done. So without loss of generality, let $f(0) > 0$.
   Claim- $f(x) < f(0) + \frac{x}{2} \forall x \in \mathbb{R}$
   Proof-
   For the sake of contradiction let $\exists x_0 > 0$ s.t $f(x_0) > f(0) + \frac{x_0}{2}$
   By LMVT, $\exists c \in (0, x_0)$ s.t
   $$f'(c) = \frac{f(x_0) - f(0)}{x_0 - 0}$$
   $$\implies f'(c) > \frac{f(0) + \frac{x_0}{2} - f(0)}{x_0} = \frac{1}{2}$$
   This is a contradiction. Our claim is proved.
   Take any $x > 2 \times f(0)$, call it x_1 that would make $f(0) + \frac{x_1}{2} < x_1$, then
   $$f(x_1) < f(0) + \frac{x_1}{2} < x_1$$
   Now we can see that $f(0) > 0$ and $f(x_1) < x_1$.
   Apply Intermediate Value Property on the function $g(x) = f(x) - x$, this is clearly continuous with $g(0) > 0$ and $g(x_1) < 0$
   We know $\exists x_0 \in (0, x_1)$ s.t $g(x_0) = x_0$.
   Thus we have x_0 such that $f(x_0) = x_0$
   Similarly if $f(0) < 0$ we get some $x_1 < 0$ and a $x_0 \in (0, x_1)$ such that this holds.

2. WLOG, $A \leq B \leq C$. Rewriting the given equation using $\cos^2 A + \sin^2 A = 1$, we get
   $$\begin{aligned} &3(\cos^2 A + \cos^2 B + \cos^2 C) = 3 \\ &\iff \cos^2 A + \cos^2 B = \sin^2(A + B) \\ &\iff \cos^2 A + \cos^2 B = \sin^2 A \cos^2 B + \sin^2 B \cos^2 A + 2 \sin A \cos A \sin B \cos B \\ &\iff 2\cos^2 A \cos^2 B = 2 \sin A \sin B \cos A \cos B \\ &\iff \sin A \sin B = \cos A \cos B, \end{aligned}$$
   which shows that $\cos(A + B) = 0 \implies A + B = \pi/2 \implies C = \pi/2$.

3. Clearly, f is a bounded function, since it is continuous and defined on a closed interval. Suppose that $|f(x)| \leq M$ for some $M > 0$.
   $$|f(x)| = |f(x) - f(0)| = \left| \int_0^x f'(t) dt \right| \leq \int_0^x |f'(t)| dt \leq \int_0^x M dt = Mx.$$
   Note that the above inequality holds for every $x \in [0, 1]$. Using this new bound Mt on $f(t)$, we

can again apply the same inequality as described above to get

$$f(x) \leq \int_0^x f(t)dt \leq M\frac{x^2}{2}.$$

Inductively, for every $n \geq 1$, $f(x) \leq M\frac{x^n}{n!} \leq \frac{M}{n!}$. Letting $n \to \infty$, we get $f(x) = 0$ for all $x \in [0,1]$.

4. Observe that, $f^{(2)}(z) = (z^2)^2 = z^4$, and by Induction, one can easily show that $f^{(n)}(z) = (f^{(n-1)}(z))^2 = (z^{2^{(n-1)}})^2 = z^{2^n}$.
Now, z has period n, will imply $z^{2^n} = z$, hence $z^{2^n-1} = 1$, i.e. $z = e^{\frac{i\pi}{2^n-1}}$.
Let $z \in S^1$, and z has period k $(k < n)$. Say, $n = qk + r$ Then,
$z^{2^n} = (z^{2^{qk}})^{2^r} = z^{2^r}$. Now $f^{(n)}(z) = z \Rightarrow z^{2^r} = z$, but $r < k$, hence $r = 0$
Hence, $k|n$. Now, if we call the number of complex numbers in S^1 with period n as a_n, we can conclude that,

$$\sum_{k|n} a_k = 2^n - 1$$

If $n = p^m$ for some prime p, $\sum_{k|p^m} a_k = \sum_{k|p^{m-1}} a_k + a_n = 2^{p^{m-1}} - 1 + a_{p^m} = 2^{p^m} - 1$, which implies $a_{p^m} = 2^{p^m} - 2^{p^{m-1}}$
Now, if $n = p^m q$ for some different primes p and q,

$$\sum_{k|p^m q} a_k = \sum_{k|p^{m-1}q} a_k + a_{p^m} + a_{p^m q} = 2^{p^{m-1}q} - 1 + 2^{p^m} - 2^{p^{m-1}} + a_{p^m q} = 2^{p^m q} - 1$$

Hence, $a_{p^m q} = 2^{p^m q} - 2^{p^{m-1}q} - 2^{p^m} + 2^{p^{m-1}}$.
Finally, if $n = p^m q^2$,

$$\sum_{k|p^m q^2} a_k = \sum_{k|p^{m-1}q^2} a_k + a_{p^m} + a_{p^m q} + a_{p^m q^2}$$

$$= 2^{p^{m-1}q^2} - 1 + 2^{p^m} - 2^{p^{m-1}} + 2^{p^m q} - 2^{p^{m-1}q} - 2^{p^m} + 2^{p^{m-1}} + a_{p^m q^2} = 2^{p^m q^2} - 1$$

Which implies, $a_{p^m q^2} = 2^{p^m q^2} - 2^{p^{m-1}q^2} - 2^{p^m q} + 2^{p^{m-1}q}$
Plugging in $p = 3, m = 4, q = 5$, we have $a_{2025} = 2^{2025} - 2^{675} - 2^{405} + 2^{135}$

5. **Given:**

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{1}{a+b+c}$$

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$$\frac{1}{a} + \frac{1}{b} = \frac{1}{a+b+c} - \frac{1}{c} \Rightarrow \frac{1}{a} + \frac{1}{b} = \frac{-(a+b)}{(a+b+c)c}$$

$$\frac{a+b}{ab} = \frac{-(a+b)}{(a+b+c)c}$$

$$(a+b)\left[\frac{1}{ab} + \frac{1}{(a+b+c)c}\right] = 0$$

Take LCM of the terms inside:

$$(a+b)\left[\frac{c(a+b+c)+ab}{abc(a+b+c)}\right]=0 \Rightarrow (a+b)\left(\frac{c^2+ac+ab+bc}{abc(a+b+c)}\right)=0$$

So,

$$\frac{(a+b)(c^2+ac+ab+bc)}{abc(a+b+c)}=0$$

Now factor the numerator:

$$(a+b)(c^2+ac+ab+bc)=(a+b)(b+c)(c+a)$$

So we get:

$$\frac{(a+b)(b+c)(c+a)}{abc(a+b+c)}=0$$

This implies:

$$(a+b)(b+c)(c+a)=0 \Rightarrow a=-b \quad \text{or} \quad b=-c \quad \text{or} \quad c=-a$$

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Let us assume $a=-b$. Then:

$$\text{LHS}=\frac{1}{a^k}+\frac{1}{b^k}+\frac{1}{c^k}=\frac{1}{a^k}+\frac{1}{(-a)^k}+\frac{1}{c^k}$$

Since k is odd, $(-a)^k=-a^k$, so:

$$\frac{1}{a^k}-\frac{1}{a^k}+\frac{1}{c^k}=\frac{1}{c^k}$$

Now, the RHS:

$$\frac{1}{(a+b+c)^k}=\frac{1}{(a-a+c)^k}=\frac{1}{c^k}$$

Hence, LHS = RHS.

Like this, we can similarly show the identity holds for the other two cases:

$$b=-c \quad \text{and} \quad c=-a$$

6. Without loss of generality, we may assume that S contains only positive integers.

   Let

   $$S=\{2^{a_i}3^{b_i} \mid a_i,b_i\in\mathbb{Z},\ a_i,b_i\geq 0,\ 1\leq i\leq 9\}.$$

   It suffices to show that there are $1\leq i_1,i_2,i_3\leq 9$ such that

   $$a_{i_1}+a_{i_2}+a_{i_3}\equiv b_{i_1}+b_{i_2}+b_{i_3}\equiv 0 \pmod 3.$$

For $n = 2^a 3^b \in S$, let's call $(a \pmod 3, b \pmod 3)$ the type of n. Then there are 9 possible types:

$$(0,0),(0,1),(0,2),(1,0),(1,1),(1,2),(2,0),(2,1),(2,2).$$

Let $N(i,j)$ be the number of integers in S of type (i,j). We obtain 3 distinct integers whose product is a perfect cube when

(1) $N(i,j) \geq 3$ for some i,j,or

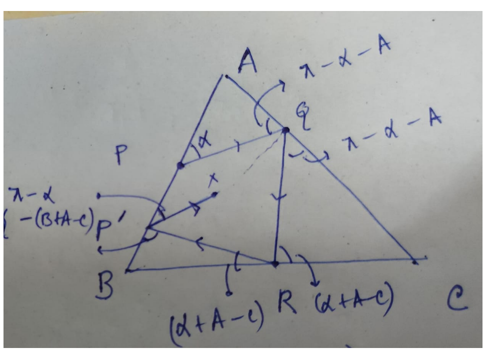
(2) $N(i,0)N(i,1)N(i,2) \neq 0$ for some $i = 0,1,2$,or

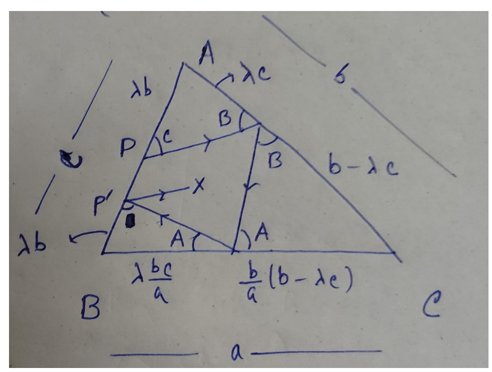
(3) $N(0,j)N(1,j)N(2,j) \neq 0$ for some $j = 0,1,2$,or

(4) $N(i_1,j_1)N(i_2,j_2)N(i_3,j_3) \neq 0$, where $\{i_1,i_2,i_3\} = \{j_1,j_2,j_3\} = \{0,1,2\}$

Assume that none of the conditions (1)∼(3) holds. Since $N(i,j) \leq 2$ for all (i,j), there are at least five $N(i,j)$'s that are nonzero. Furthermore, among those nonzero $N(i,j)'$s, no three have the same i nor the same j. Using these facts, one may easily conclude that the condition (4) should hold. (For example, if one places each nonzero $N(i,j)$ in the (i,j)-th box of a regular 3×3 array of boxes whose rows and columns are indexed by 0,1 and 2, then one can always find three boxes, occupied by at least one nonzero $N(i,j)$, whose rows and columns are all distinct. This implies (4).)

7. Consider the angles of ΔABC as $\angle A, \angle B, \angle C$ and the sides opposite to this angles a, b, c respectively.





Let the ball starts from a point P on the side AB with an angle α with AB and hit the side AC at point Q and then hit the side BC at point R and then once again hit the side AB at point P'. Then from ΔPAQ, $\angle AQP = \pi - \alpha - A$. Angle of incidence at $Q = \pi/2 - (\pi - \alpha - A)$. So angle of reflection at $Q = \pi/2 - (\pi - \alpha - A)$,since angle of incidence = angle of reflection. Then $\angle CQR = \pi/2-$ (angle of reflection at Q)$= \pi - \alpha - A$. From $\Delta CQR, \angle CRQ = \pi - \angle CQR - C = A + \alpha - C$. We can show that for the reflection at R, $\angle BRP' = \angle CRQ = A + \alpha - C$. In $\Delta BRP', \angle BP'R = \pi - B - \angle BRP' = \pi - \alpha - (B + A - C)$. Let the reflected ray from point P' is X. Again from reflection at P', $\angle AP'X = \angle BP'R = \pi - \alpha - (B + A - C)$. For a triangular path we want $\angle QPA = \angle AP'X \implies \alpha = \angle C$.

If we start the ball at an angle $\angle C$ with side AB, then $\Delta APQ \sim \Delta ABC$. Let $AP = \lambda b$, then by similarity $AQ = \lambda c$. So $CQ = b - \lambda c$. Again $\Delta CQR \sim \Delta ABC$, so $CR = \frac{b}{a}(b - \lambda c)$. Then $BR = a - CR = \lambda \frac{bc}{a}$. $\Delta BQP' \sim \Delta ABC$, so $BP' = \lambda b$. Since we want $P = P'$, this implies

$BP' + PA = BA \implies 2\lambda b = c \implies \lambda = \frac{c}{2b}$.
So if the starting point of the ball is point P (with starting ray PX) on AB such that $AP = AB/2$ and $\angle APX = \angle ACB$ then there will be a triangular path for the ball.

8. First we show that for any positive integers $x_1, \ldots, x_n$, we have $(x_1+1)\cdots(x_n+1) \geq 2(x_1+\cdots+x_n)$ using induction on n. Suppose that the first k many a_i's are ones and every term beyond that is > 1. Then,

$$a_1 a_2 \ldots a_n = a_{k+1}a_{k+2}\cdots a_n \geq 2\left((a_{k+1}-1)+(a_{k+2}-1)+\cdots+(a_n-1)\right) = 2a_1a_2\cdots a_n - 2n.$$

This completes the proof. For equality, say $a_{n-1} = a, a_n = b$. Then $n-2+a+b = ab \Rightarrow (a-1)(b-1) = n-1$. Putting $a+b = n+2 \Rightarrow a-1+b-1 = n$, we can say $a-1 = 1$, and hence only equality case is $\{1, \ldots 1, 2, n\}$

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